Quantum Fluctuations in Microwave Radiometry*

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Summary—This paper assesses the possible significance of the quantum nature of electromagnetic radiation in limiting the measurement accuracy attainable with a microwave radiometer. Analogies are shown to exist between the form of a formula describing fluctuations in the radiometer output, and both a formula describing the radiometer input signal, and also, a formula describing the output of a photocell detector. Detailed quantum mechanical consideration of the processes of amplification and detection are circumvented by considering how the formula for fluctuations in the radiometer output might be modified so as to make it consistent with the measurement precision implied by these other formulas. A modified formula is suggested which includes a quantum fluctuation whose magnitude depends on signal power.

I. INTRODUCTION

THE development of microwave radiometers with very low noise figures and very wide bandwidths has been steadily decreasing the average number of photons per unit time-bandwidth represented by the minimum perceptible signal. It is, therefore, of interest to consider whether the quantum nature of electromagnetic radiation might be significant in limiting the measurement accuracy attainable in microwave radiometry.

Weber[1] and Strandberg[2] have computed the ultimate limit on sensitivity set by spontaneous emission noise for both maser and vacuum tube amplifiers. The equivalent temperature of this noise is only hf/k for a maser and $hf/2k$ for a vacuum tube amplifier. Perhaps of more immediate significance for microwave radiometry is the possibility of a quantum fluctuation dependent on signal power analogous to the random counting rate of a photon counter. Gabor[3,4] has demonstrated mathematically that it is not possible to extract more information from a signal with electron tube apparatus capable of measuring both amplitude and phase than with photon counters. Accordingly, quantum mechanics should place similar restrictions on the degree to which fluctuations can be reduced in optical and microwave radiometry. Recently, wave and particle fluctuations have been measured simultaneously in the outputs of optical photocell detectors.[5,6] In Section II, we shall show that wave and particle fluctuations also coexist in the description statistical mechanics gives for the input signal to a microwave radiometer. In Section III, we show that the usual formula for the fluctuations in the output of a microwave radiometer has an analogous form to the portion associated with wave effects of both the formula found in Section II and also a formula describing the wave and particle fluctuations in the output of a photocell detector. In Section IV, we show how the microwave radiometer formula might be modified to include a particle fluctuation term of the form suggested by these analogies. In Section V, we compare the magnitude of this particle fluctuation with that of the classical wave fluctuation.

II. FLUCTUATIONS IN A RADIO SIGNAL

In this section, we shall deduce a formula describing fluctuations in the input signal of a wide-band microwave radiometer. If the input signal consists of thermal radiation, it may be described by the Planck radiation formula. In its most familiar form the Planck radiation formula is written as the product of two factors. One factor, $8\pi f^2 V df/c^3$, specifies the number of standing-wave modes between frequencies f and $f+df$ in a large, three-dimensional volume V. The other factor, $hf/(\exp hf/kT-1)$, gives the energy associated with each mode. For a long one-dimensional microwave transmission line of length L, for which only one field polarization is possible, the first factor becomes instead $(2L/c)df$. This gives it the frequency independence necessary for radiative equilibrium with the Johnson noise of a resistive termination. In the language of statistical mechanics, each standing-wave mode corresponds to a quantum state, and the Bose-Einstein distribution $1/(\exp hf/kT-1)$ is regarded as specifying the number of quanta of energy hf in the quantum state associated with the mode of frequency f. Our interest will be in the time fluctuation which the Bose-Einstein probability distribution implies in the number of quanta occupying a group of adjacent quantum states. From a communication theory viewpoint these fluctuations in numbers of quanta constitute energy fluctuations within the frequency bandwidth corresponding to these quantum states.

This fluctuation may be computed by quantum statistics. We shall describe the relationship of this theory to electromagnetic theory only in sufficient detail to permit stating the computation of the fluctuation as given by Landau and Lifschitz[7] who give an extensive discussion of fluctuations in statistical mechanics. In the classical Rayleigh-Jeans wave theory of thermal

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[1] J. Weber, "Maser noise considerations," *Phys. Rev.*, vol. 108, pp. 537–541; November 1, 1957.

[2] M. W. P. Strandberg, "Inherent noise of quantum mechanical amplifiers," *Phys. Rev.*, vol. 106, pp. 617–620; May 15, 1957.

[3] D. Gabor, "Communication theory and physics," *Phil. Mag.*, vol. 41, pp. 1161–1187; November, 1950.

[4] D. Gabor, "Der Nachrichtengehalt eines elektromagnetischen Signals," *Archiv der Elektrischen Ubertragung*, vol. I, pp. 95–99; February, 1953.

[5] R. Q. Twiss and A. G. Little, "The detection of time-correlated photons by a coincidence counter," *Australian J. Phys.*, vol. 12, pp. 77–93; January, 1959.

[6] M. O. Harwit, "Measurement of Radiation Fluctuations from a Source in Thermal Equilibrium," Research Lab. of Electronics, M.I.T., Cambridge, Quart. Prog. Rept., pp. 23–26; April 15, 1960.

[7] L. D. Landau and E. M. Lifschitz, "Statistical Physics," Addison-Wesley Publishing Co., Inc., Reading, Mass.; 1958. See especially p. 358. (Translated from the Russian by E. and R. F. Peierls.)

radiation, equal energy was accorded each standing-wave mode. In the quantum statistical description, the total energy is assumed distributed among all the modes in the most probable way consistent with the constraint that the energy of each mode be an integral multiple of hf. This gives particle properties to the radiation since the computation parallels the classical particle mechanics computation of the most probable distribution of gas particles among cells in phase space, such that all particles in a given cell have equal kinetic energy. In fact, the only essential difference is the lack of an equivalent for the possibility which exists with classical particles of imagining them individually labelled so that one can speak of which particle is in which cell. This has no equivalent in the quantum statistical theory where only the total number of quanta associated with each standing-wave mode or quantum state need be specified to completely describe a possible energy distribution. The quantum statistical theory may thus be regarded alternatively as describing waves with quantized energies or, equivalently, particles which are intrinsically indistinguishable from one another.

This intrinsic indistinguishability of particles must be assumed for gas particles, in general, in order to obtain a correct quantum mechanical description, and so is not a peculiar property of massless photons. The photon distribution formula $1/(\exp hf/kT-1)$ may thus be regarded as a special case of the general Bose-Einstein distribution formula applicable to gas particles not subject to the Pauli exclusion principle. This formula may also be used to compute the mean number of particles in a single quantum state, in which case it may be written as

$$\overline{n_s} = \frac{1}{e^{(E_s-\mu)/kT} - 1} \tag{1}$$

where E_s is the kinetic energy, equal to hf for photons, and μ is the chemical potential, equal to zero for photons.

Since this mean is associated with a thermal equilibrium, the precise number of particles in the state will fluctuate about $\overline{n_s}$ with time. The mean-square fluctuation may be computed by the general formula[7]

$$\overline{(\Delta n_s)^2} = kT \frac{\partial \overline{n_s}}{\partial \mu} \tag{2}$$

which describes the fluctuation in number of particles in a portion of a gas at constant temperature and volume. Applying this to (1) gives

$$\overline{(\Delta n_s)^2} = \frac{e^{(E_s-\mu)/kT}}{(e^{(E_s-\mu)/kT} - 1)^2} = \overline{n_s} + \bar{n}_s^2 \tag{3}$$

for all Bose-Einstein particles and for photons in particular. The statistical independence of the quanta permits summing over a group of N adjacent quantum states containing altogether $n = \Sigma n_s$ quanta to obtain for the mean-square fluctuation in the number of quanta associated with N adjacent quantum states

$$\overline{(\Delta n)^2} = N(\overline{n_s} + \bar{n}_s^2) = N\overline{n_s} + \frac{(N\overline{n_s})^2}{N} = \bar{n} + \frac{\bar{n}^2}{N}. \tag{4}$$

The corresponding fluctuation in energy E obtained by multiplying by $(hf)^2$ is

$$\overline{(\Delta E)^2} = hf\overline{E} + \overline{E}^2/N. \tag{5}$$

This expression is interesting because, as first shown by Einstein and Lorentz,[8] it gives the total fluctuation as the sum of two terms which separately correspond to either classical particle or classical wave phenomena. The first term by itself is the $\sqrt{\bar{n}}$ rms fluctuation kinetic theory gives for the fluctuation in the number of classical particles in a small region of a large gas. The second term by itself is identifiable with the rms energy fluctuation proportional to squared amplitude which results from beats between random classical waves and is normally the only fluctuation considered in communication theory.

In order to evaluate the number of quantum states N associated with a portion of the radiometer input signal, we need a correspondence between the equilibrium quantum states we have associated with the Planck radiation law and the received signal. The needed correspondence is implicit in the previous statement that the number of standing-wave modes or quantum states between frequencies f and $f+df$ in a line of length L is $(2L/c)\Delta f$. The length of line per quantum state obtained by dividing the number of states into L is $\Delta x = 2c/\Delta f$. Halving the number of degrees of freedom to correspond to the fact that we are interested in propagation in only one direction gives $\Delta x \Delta f = c$. In terms of the particle momentum $p = hf/c$, this is equivalent to the usual association of quantum states with cells of area $\Delta x \Delta p = h$ in two-dimensional phase space. Alternatively, since a wave propagated at velocity c advances a distance $c\Delta t = \Delta x$ in time Δt, one may say that quantum states correspond to cells of area $\Delta t \Delta f = 1$ on the time-frequency or information plane. This correspondence was first introduced by Gabor[3] and has been used by Stern[9] to analyze the information handling capabilities of a discrete photon channel. It permits one to say that the number of quantum states in a duration Δt of signal extending over a frequency range Δf is the

[8] H. A. Lorentz, "Les Theories Statistiques en Thermodynamic," Teubner Verlag, Leipzig and Berlin, Germany; 1916.

[9] T. E. Stern, "Some quantum effects in information channels," IRE Trans. on Information Theory, vol. IT-6, pp. 435–440; September, 1960.

number of cells of unit area in the rectangle on the information plane representing the time-bandwidth product, $\Delta t\Delta f$.

A microwave radiometer accepts only the signal power in a bandwidth Δf set by its reception filter and averages this power over an integrating time Δt set by its final smoothing filter. Accordingly, the portion of input signal energy of significance in determining fluctuations may be regarded as that included in the time-bandwidth $\Delta t\Delta f$. Since this time-bandwidth product also equals the number of quantum states in the corresponding portion of signal, it may be taken as the appropriate value of N in (4) and (5). Thus, the mean-square energy fluctuation or equivalently the mean-square fluctuation in power integrated over time Δt in the radiometer input signal is:

$$\overline{(\Delta E)^2} = hf\overline{E} + \overline{E}^2/\Delta t\Delta f, \tag{6}$$

and the mean-square fluctuation in the quanta contained in this portion of the signal is

$$\overline{(\Delta n)^2} = \bar{n} + \frac{\bar{n}^2}{\Delta t\Delta f}\cdot \tag{7}$$

The first term in these formulas is a consequence of the quantum nature of the radiation and has no equivalent in classical wave theory. The second term corresponds to the usual wave fluctuation associated with a random radio signal. In Appendix II we show that the fluctuation in the energy of random waves along a transmission line is given by a formula of this form. Rice[10,11] has computed the mean-square fluctuation in a random current. The formula he obtains for the mean-square fluctuation is of similar form to (29) of Appendix II and, as explained there, it may be transformed into the form $\overline{E}^2/\Delta t\Delta f$ by a change of variables.

III. Analogies Between Fluctuation Formulas

Purcell[12] and Mandel[13] find for the mean-square fluctuation in the average number of photoelectrons produced in time Δt in a photocell illuminated by Gaussian random waves from a spectral line of width Δf

$$\overline{(\Delta n)^2} = \bar{n} + \alpha\frac{\bar{n}^2}{\Delta t\Delta f}\cdot \tag{8}$$

The first term corresponds to the usual $\sqrt{\bar{n}}$ rms fluctuation in counting rate associated with all particle counters. The second term results from the added assumption of wave fluctuations in the light illuminating the photocell. α is a constant of the order of unity. Since a similar constant would be required in (7), had we not tacitly assumed integrating time and bandwidth definitions such as to make the minimum Fourier integral uncertainty product $\Delta t\Delta f$ precisely unity, the form of (8) may be regarded as similar to that of (7).

The usual expression for the mean-square fluctuation in the output of a microwave radiometer may be written in the general form[14]

$$\overline{(\Delta z)^2} = \frac{\bar{z}^2}{\Delta t\Delta f} \tag{9}$$

where $\bar{z}$ denotes the mean output meter deflection, Δf the reception filter bandwidth, and Δt the smoothing filter integrating time. $\sqrt{\overline{(\Delta z)^2}}$ is the rms fluctuation in output meter deflection occasioned by the random character of both signal and system noise.

As in (7), the absence of a factor of proportionality of the order of unity in (9) depends on the choice of definitions for integrating time and bandwidth. Dicke,[15] for example, gives this proportionality factor as $\pi^{3/2}/8$. The significant point for the qualitative discussion intended here is that this formula for the fluctuation at the receiver output is of similar form to the terms representing classical wave effects in both (6), which describes fluctuations at the receiver input, and (8), which describes fluctuations in the output of a photoelectric detector.

IV. Modification of Radiometer Formula

In this section, we shall show how (9) might be modified so as to include a quantum fluctuation term. For a receiver which contributes no noise of its own, the mean deflection $\bar{z}$ is directly proportional to signal power. For this limiting case, (9), like (6), (7), and (8) may be regarded as representing signal fluctuations. However, as stated in Section I, Gabor has shown that no greater measurement accuracy should be possible with radio equipment than for the measurements to which (6), (7), and (8) apply. Formula (9) differs in form from these formulas only by the lack of a term associated with particle fluctuations. Formula (7) may be written as

$$\overline{(\Delta n)^2} = \left(\frac{\Delta t\Delta f}{\bar{n}} + 1\right)\frac{\bar{n}^2}{\Delta t\Delta f}\cdot \tag{10}$$

Thus, a modified fluctuation formula for a microwave radiometer with unity noise figure equivalent to (9) at

[10] S. O. Rice, "Mathematical analysis of random noise," in "Selected Papers on Noise and Stochastic Processes," N. Wax, Ed., Dover Publications, Inc., New York, N. Y., p. 227; 1954. (Reprinted from *Bell Sys. Tech. J.*, vol. 23, pp. 282–322, July, 1944; vol. 24, pp. 47–156, January, 1945.)

[11] S. O. Rice, "Filtered thermal noise-fluctuation of energy as a function of interval length," *J. Acoust. Soc. Am.*, vol. 14, pp. 216–227; April, 1943.

[12] E. M. Purcell, "The question of correlation between photons in coherent light rays," *Nature*, vol. 178, pp. 1449–1450; December 29, 1956.

[13] L. Mandel, "Fluctuations of photon beams and their correlations," *Proc. Phys. Soc.*, vol. 71, pp. 1037–1048; December, 1958.

[14] R. S. Colvin, "Faint signal limitations of radiometers," 1959 IRE WESCON Convention Record, pt. 8, pp. 52–58.

[15] R. H. Dicke, "The measurement of thermal radiation at microwave frequencies," *Rev. Sci. Instr.*, vol. 17, pp. 268–275; July, 1946.

high intensities, and of similar form to (6), (7), and (8) is

$$\overline{(\Delta z)^2} = \left(\frac{\Delta t \Delta f}{\bar{n}} + 1\right) \frac{\bar{z}^2}{\Delta t \Delta f} \tag{11}$$

where $\bar{n}$ denotes the mean number of photons received in the integrating time Δt.

The assumption of an ideal receiver which adds negligible noise to the signal prevents the usual interpretation of the mean-square fluctuation given by (9) as the minimum perceptible signal. This usual interpretation is based on the opposite assumption that for a weak signal the output meter deflection is mainly due to receiver noise. Then the fluctuation in meter deflection is primarily a fluctuation in system noise, indistinguishable from an incremental deflection representing a weak signal unless it exceeds this fluctuation. In the limit considered here of no receiver noise, the mean output meter deflection is proportional to mean signal strength alone and the fluctuation in deflection is a fluctuation in measured signal strength. This fluctuation does determine the accuracy with which the mean signal strength can be measured. It does not, however, limit the minimum signal which can be detected, since the mean-square fluctuation decreases with signal strength according to either the unmodified formula (9), or the modified formula (11).

Gabor[4] concludes that quantum mechanics prevents measuring an electromagnetic signal in steps smaller than the square root of the number of quanta it contains with electron tube apparatus. The mean-square fluctuation given by (11) differs from the purely wave fluctuation of (9) by an amount $\bar{z}^2/\bar{n}$. The corresponding rms relative fluctuation $\sqrt{\overline{(\Delta z)^2}}/\bar{z}$ equals $1/\sqrt{\bar{n}}$. Thus, the quantum fluctuation in (11) is the minimum fluctuation consistent with Gabor's general result.

V. Comparison of Wave and Particle Fluctuations

Since the rms particle fluctuation may be regarded as a fractional fluctuation of $1/\sqrt{\bar{n}}$ in signal power, its absolute magnitude increases with signal power. Whether it is likely to be of consequence in a particular radiometer measurement is, therefore, likely to depend on its size relative to the wave fluctuation whose magnitude increases even more rapidly with signal power.

The ratio of the mean-square wave fluctuation to the mean-square particle fluctuation in (11) equals the mean number of quanta per quantum state. Thus, if this ratio is denoted as R,

$$R = \bar{n}/\Delta t \Delta f. \tag{12}$$

If $\bar{n}$ is estimated as signal power P times integrating time Δt divided by the mean quantum energy expressed as the product of Planck's constant h and the mean signal frequency $\bar{f}$, so that $\bar{n} = (P\Delta t)/h\bar{f}$, then the ratio of the mean-square fluctuations may be written in the equivalent form

$$R = \frac{P\Delta t}{h\bar{f}} \div \Delta t \Delta f = \frac{P}{h\bar{f}\Delta f} \cdot \tag{13}$$

$h\bar{f}\Delta f$ is the minimum sensitivity set by spontaneous emission noise for a maser amplifier since the equivalent temperature[2] of this noise $T = hf/k$ corresponds to a minimum power $kT\Delta f = h\bar{f}\Delta f$. From this, it is apparent that these formulas can have no significance unless R is greater than unity. Such a limit is to be expected since many particle statistics were used to derive (4).

VI. Conclusions

Analogies have been shown to exist between the dependence of output meter deflection fluctuations of a microwave radiometer on bandwidth and integrating time, and the term representing wave fluctuations in two formulas which include quantum effects and describe, respectively, fluctuations in a radiometer input signal and fluctuations in a photocell output current. It is shown that a particle fluctuation dependent on signal power results if the usual formula for the mean-square fluctuation in the output of a microwave radiometer with unity noise figure is modified to have the same form as these other formulas. Although the form of the modified formula is inferred by analogy, the modification is required by Gabor's general theorem which denies the possibility of greater signal measurement accuracy with electron tube apparatus than with particle counters.

Appendix I

Effect of Antenna

In this appendix, we shall consider how the operation of the antenna system of the receiver is related to the fluctuations.

Purely wave considerations are adequate to describe the coupling which an antenna effects between degrees of freedom in three-dimensional space and a one-dimensional transmission line. In fact, if the antenna did not couple equal numbers of degrees of freedom, a line terminated by a matched load at one end and at the other by an antenna in an enclosure of the same temperature as the line and matched load, could not be in thermal equilibrium simultaneously at all frequencies.[15] Thus, in Section II, we were able to compute the number of degrees of freedom involved by simply counting the number of possible standing-wave modes for the transmission line. Since the statistical mechanics computation required only a knowledge of this number, specific consideration of the antenna was unnecessary.

It may not be entirely evident that this computation of degrees of freedom using purely wave concepts is adequate to account for the particle as well as the wave nature of the radiation. In order to clarify this fact, we shall give an alternative computation based on particle considerations. This computation leads to the same association of quantum states with information cells of area $\Delta t\Delta f=1$ as we found using only wave concepts. This shows that the particle nature of the radiation in no way modifies the result and that classical wave theory is completely adequate to describe the guiding of the radiation into the transmission line by the antenna. In as much as from a wave theory viewpoint the antenna may be regarded as a suitable aperture to create a desired diffraction pattern, it is interesting to compare this situation with X-ray diffraction. The X-ray diffraction pattern created by a calcite crystal can similarly be computed from purely wave considerations even when the intensity is so weak that the photons "guided by" the diffraction pattern are counted only one at a time.

We now consider how the coupling of the radiation field in space to a transmission line may be considered using particle statistics. The radiation in three-dimensional space will be considered to consist of a photon gas. As is usual in gas theory, quantum states will be associated with cells of volume h^3 in the phase space associated with the gas particles which are in this case photons, and where h is Planck's constant. In order to determine the portion of signal corresponding to one quantum state, we shall relate the volume of one of these cells in the six-dimensional phase space associated with three-dimensional real space to the area of a corresponding cell in the two-dimensional phase space associated with the one-dimensional antenna feed line.

To do this, we note that an antenna with an effective aperature of width a will have a beam angle of approximately λ/a radians were λ signifies the wavelength. From a particle point of view, one says instead that there is an uncertainty in the direction of arrival of a photon. If the usual momentum $p=hf/c$ is associated with a photon, the corresponding uncertainty in either transverse components of momentum is $(hf/c)\sin(\lambda/a)$ or for small angles approximately $(hf/c)(\lambda/a)=h/a$, corresponding to the Heisenberg uncertainty relation $\Delta p\Delta x\geq h$ with $\Delta x=a$. A similar uncertainty in the forward component of momentum of magnitude $h\Delta f/c$ might result from a receiver bandwidth Δf. Thus, since the volume of a cell in phase space is h^3 we have

$$\begin{aligned} h^3 &= \Delta x\Delta y\Delta z\Delta p_x\Delta p_y\Delta p_z \\ &= \Delta x\Delta y\Delta z(h/a)(h/a)(h\Delta f/c) \end{aligned} \tag{14}$$

making the volume in real space corresponding to a cell in phase space

$$\Delta x\Delta y\Delta z = \frac{a^2c}{\Delta f}\,. \tag{15}$$

Putting $\Delta x=\Delta y=a$ gives the length in the antenna feed line corresponding to one cell as

$$\Delta z = c/\Delta f. \tag{16}$$

In time Δt, the radiation will travel a distance $\Delta z=c\Delta t$, if the transmission line has propagation velocity c. Hence, we can also say that the quanta received in a time

$$\Delta t = \frac{1}{\Delta f} \tag{17}$$

may be regarded as belonging to the same phase-space cell. The portion of signal in a one-dimensional transmission line which the antenna associates with a cell in six-dimensional phase space may thus be associated alternatively with cells of area $\Delta z\Delta p_z=(c/\Delta f)(h\Delta f/c)=h$ in the two-dimensional phase space of the photons in the transmission line or cells of area $\Delta t\Delta f=1$ in a two-dimensional time-frequency or information plane. The precise values of these uncertainty products, of course depend on the definitions used for angular beamwidth, frequency bandwidth, and effective antenna aperture.

Appendix II

Wave Fluctuation in a Transmission Line

In this appendix, we compute an appropriate wave-theory formula for the mean-square energy fluctuation in a one-dimensional black body consisting of a transmission line with matched terminations at both ends such as used, for example, in the Nyquist derivation of the Johnson noise formula. The method of derivation and notation will be chosen to emphasize the similarity of this computation both to Lorentz's computation for the equivalent three-dimensional case[8] and to Rice's computation of the fluctuation in a random current.[10,11]

We assume that the current along the line has statistical properties permitting the representation

$$I_N(z,t) = \sum_{n=-N}^{N} C_n \cos(\omega_n t + \beta_n z + \phi_n) \tag{18}$$

where the ϕ_n are random angles. If the fundamental radian frequency is denoted by $\Delta\omega$ and the constant phase velocity by c, $\omega_n=|n|\Delta\omega$ and $\beta_n=n(\Delta\omega/c)=n\Delta\beta$. That is, frequency is always considered positive, but positive and negative wave numbers are associated respectively with waves traveling in negative and positive directions. We let C_n equal the actual current amplitude multiplied by the square root of the inductance per unit length so that C_n^2 is the average energy per unit line length in the electromagnetic fields of a single traveling wave.

The energy per unit line length is then

$$\begin{aligned} I_N^2(z,t) = &\sum_{n=-N}^{N} C_n^2\cos^2(\omega_n t + \beta_n z + \phi_n) \\ &+ 2\sum_{n=-N}^{N-1}\sum_{m=n+1}^{N} C_nC_m\cos(\omega_n t + \beta_n z + \phi_n) \\ &\cdot\cos(\omega_m t + \beta_m z + \phi_m). \end{aligned} \tag{19}$$

Integrating over z gives the energy in a length L of the line as

$$E = \int_{-L/2}^{L/2} I_N^2(z, t)dz$$

$$= L\left\{\frac{1}{2}\sum_{n=-N}^{N} C_n^2 + \frac{1}{2}\sum_{n=-N}^{N} C_n^2 P_n \cos(2\omega_n t + \phi_n)\right.$$

$$+ \sum_{n=-N}^{N-1}\sum_{m=n+1}^{N} Q_{nm}C_nC_m \cos[(\omega_n + \omega_m)t + (\phi_n + \phi_m)]$$

$$+ \sum_{n=-N}^{N-1}\sum_{m=n+1}^{N} R_{nm}C_nC_m \cos[(\omega_n - \omega_m)t$$

$$\left. + (\phi_n - \phi_m)]\right\}, \quad (20)$$

where

$$P_n = \frac{\sin\beta_n L}{\beta_n L}, \quad Q_{nm} = \frac{\sin[(\beta_n + \beta_m)(L/2)]}{(L/2)(\beta_n + \beta_m)},$$

$$R_{nm} = \frac{\sin[(\beta_n - \beta_m)(L/2)]}{(L/2)(\beta_n - \beta_m)}.$$

We shall compute the mean energy in length L and its mean-square fluctuation by averaging over-all possible values of the ϕ_n. These averages may be easily computed using the fact that a cosine function with random phase averages zero and also that its square averages $\frac{1}{2}$.

The mean energy is given by the first term

$$\overline{E} = (1/2)L\sum_{n=-N}^{N} C_n^2. \quad (21)$$

The remaining terms give the difference ϵ between the mean energy and the total energy. Since this difference accounts for the fluctuation we wish to compute, we can compute the mean-square fluctuation in energy by calculating the mean value of the square of these remaining terms.

This computation is facilitated by the following considerations. With the aid of trigonometric identities all of the cross-product terms may be readily shown to average zero. We can neglect the sum containing P since if N is large, the sums containing Q and R will contain many more terms. If we also replace the cosine squared terms by their average values we have left

$$\overline{\epsilon^2} = L^2\sum_{n=-N}^{N-1}\sum_{m=n+1}^{N}\frac{C_n^2C_m^2}{2}[Q_{nm}^2 + R_{nm}^2]. \quad (22)$$

Before completing the computation, it is convenient to replace the summations by equivalent integrations. A suitable representation may be obtained by writing the mean energy as an equivalent integration over a continuous energy spectrum

$$\overline{E} = (1/2)L\sum_{n=-N}^{N} C_n^2$$

$$= L\sum_{n=-N}^{N} W(\beta_n)\Delta\beta \to L\int_{-\infty}^{\infty} W(\beta)d\beta. \quad (23)$$

In order to effect a corresponding transformation of the expression for $\overline{\epsilon^2}$, we note that including also the N terms with $n=m$ represents a negligible percentage increase in the total number of terms which permits us to write

$$\overline{\epsilon^2} = L^2\sum_{n=-N}^{N}\sum_{m=-N}^{N}\frac{C_n^2C_m^2}{4}[Q_{nm}^2 + R_{nm}^2], \quad (24)$$

which transforms into

$$\overline{\epsilon^2} = L^2\int_{-\infty}^{\infty} W(\beta)\int_{-\infty}^{\infty} W(\beta')\left(\frac{\sin[(\beta+\beta')(L/2)]}{(\beta+\beta')(L/2)}\right)^2 d\beta' d\beta$$

$$+ L^2\int_{-\infty}^{\infty} W(\beta)\int_{-\infty}^{\infty} W(\beta')\left(\frac{\sin[(\beta-\beta')(L/2)]}{(\beta-\beta')(L/2)}\right)^2 d\beta' d\beta. \quad (25)$$

The first integrand will have an appreciable value only if β' is nearly equal to $-\beta$. The second integrand will have an appreciable value only if β' is nearly equal to β. Thus, we can make the substitutions $W(\beta') = W(-\beta)$ and $W(\beta') = W(\beta)$ in the respective integrands obtaining

$$\overline{\epsilon^2} = L^2\int_{-\infty}^{\infty} W(\beta)W(-\beta)\int_{-\infty}^{\infty}\left(\frac{\sin[(\beta+\beta')(L/2)]}{(\beta+\beta')(L/2)}\right)^2 d\beta' d\beta$$

$$+ L^2\int_{-\infty}^{\infty} W^2(\beta)\int_{-\infty}^{\infty}\left(\frac{\sin[(\beta-\beta')(L/2)]}{(\beta-\beta')(L/2)}\right)^2 d\beta' d\beta. \quad (26)$$

Each integral over β' equals $2\pi/L$ and the bidirectional symmetry of the problem makes $W(\beta) = W(-\beta)$. Thus,

$$\overline{\epsilon^2} = L^2\int_{-\infty}^{\infty} W^2(\beta)\left[\frac{2\pi}{L} + \frac{2\pi}{L}\right]d\beta = 4\pi L\int_{-\infty}^{\infty} W^2(\beta)d\beta. \quad (27)$$

Since $w(\beta) = w(-\beta)$, the integrals in the expressions for both $\overline{E}$ and $\overline{\epsilon^2}$ may be written as twice integrals from zero to infinity. We can then make the substitutions $\beta = 2\pi f/c$ valid for $\beta > 0$ and $4\pi W(\beta) = w(f)$. Defining $w(f)$ as 4π times $W(\beta)$ gives $w(f)$ the significance of energy per cycle for $0 < f < \infty$ since $W(\beta)$ signifies energy per radian for $-\infty < \beta < \infty$. Formulas (23) and (27) then become

$$\overline{E} = \frac{L}{c}\int_0^{\infty} w(f)df \quad\text{and}\quad \overline{\epsilon^2} = \frac{L}{c}\int_0^{\infty} w^2(f)df. \quad (28)$$

If we are interested in these mean values for only a limited frequency range $f_a < f < f_b$, write $T = L/c$ for the time required for a wave to propagate the length L of the line, and assume that the spectral density function $w(f)$ has a uniform value w_0 between f_a and f_b (as it does by even the quantum mechanical form of the Nyquist noise formula up to nearly the highest microwave frequencies in current use):

$$\overline{E} = Tw_0(f_b - f_a) \quad \text{and} \quad \overline{\epsilon^2} = Tw_0^2(f_b - f_a). \tag{29}$$

Eliminating w_0 by substituting the first of these formulas into the second and redesignating T as Δt and $f_b - f_a$ as Δf gives

$$\overline{\epsilon^2} = \frac{\overline{E}^2}{\Delta t \Delta f}. \tag{30}$$

Rice[10,11] obtains for the mean energy dissipated in a one-ohm resistor by a noise current with uniform spectral density w_0 during time T in bandwidth $f_b - f_a$

$$\overline{E} = Tw_0(f_b - f_a), \tag{31}$$

and for the mean-square energy fluctuation

$$\overline{\sigma_T^2} = w_0^2 T(f_b - f_a). \tag{32}$$

By similarly eliminating w_0 between these formulas, this mean-square fluctuation formula can also be converted to the characteristic form $\overline{E}^2/\Delta t \Delta f$.

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On the Resolution of a Class of Waveguide Discontinuity Problems by the Use of Singular Integral Equations*

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Summary—**It is shown that a considerable number of solutions of rectangular waveguide problems appearing in the literature are all special cases of a general treatment focused around the known solution of a singular integral equation. In terms of this a number of typical results are re-examined. The method is then applied to four new configurations, and the range of application and the limitations are examined.**

I. INTRODUCTION

THE number of waveguide problems capable of exact solution is limited to a few very simple shapes, even when the common approximations of ideal geometry and infinite wall conductivity are made. A class of problems recently amenable to exact treatment has involved configurations in which the discontinuity has separated the space into two uniform regions, $z < 0$ and $z > 0$. Examples are the radiation into free space of a semi-infinite length of guide, a bifurcation of the waveguide, and, exceptionally, a diaphragm halfway across the guide. The solutions involve the setting up of an integral equation for the field along the guide axis, or some other equivalent axis, the integral equation taking a different form on either side of the discontinuity. It is then solved by the Wiener-Hopf technique, the waveguide parameters being readily obtainable from the solution.

This method gives a rigorous result for the limited number of configurations to which it can be applied. It is not successful, however, in the majority of those cases in which the discontinuity takes the form of a variation over the cross section of the waveguide, such as, for example, diaphragms, strips, change of guide cross section, etc. Nor is it applicable to configurations in which the propagation medium changes at the discontinuity, *e.g.*, if there is a dielectric or ferrite insert.

For such cases it is more satisfactory to take the field over the cross section as the unknown variable, and a different type of integral equation can be set up for this class of problems. The Wiener-Hopf technique is no longer usuable, but the equation can be solved to various quasi-static degrees of approximation in some particular cases. This has been done by Schwinger and co-authors[1] for waveguide diaphragms, and by Lewin[2,3] for un-

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